Atmospheric turbulence effects on a partially coherent Gaussian beam: implications for free-space laser communication

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A partially coherent quasi-monochromatic Gaussian laser beam propagating in atmospheric turbulence is examined by using a derived analytic expression for the cross-spectral density function. Expressions for average intensity, beam size, phase front radius of curvature, and wave-front coherence length are obtained from the cross-spectral density function. These results provide a model for a free-space laser transmitter with a phase diffuser used to reduce pointing errors.

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1. INTRODUCTION

Atmospheric turbulence due to random variations in the refractive index of the atmospheric channel significantly limits the range and the performance of free-space laser communication systems. Atmospheric turbulence-induced random distortions of the optical phase result in intensity fading at the receiver, giving rise to system bit error rates potentially orders of magnitude higher than those in the absence of turbulence. There has been growing interest in developing techniques to overcome the turbulence-induced intensity fades that cause these communication system bit error rates. Techniques investigated to date include using multiple transmitter apertures,[1,2] adaptively controlling the transmitter beam size,[3] and utilizing adaptive optics techniques.[4,5] Nonetheless, an optimal solution to the problem of turbulence-induced intensity fading has yet to be identified. Multiple-aperture systems are large and bulky, and the electronics can be unnecessarily complex. In the presence of the strong intensity fluctuations characteristic of ground-to-ground propagation paths, traditional adaptive optics techniques are not effective because of difficulties in obtaining wave-front measurements.[6] New approaches to the problem of reducing turbulence-induced intensity fading in laser communication systems are still of much interest.

Here we propose an approach where the spatial coherence of the signal-carrying laser beam is partially destroyed before it is launched into the atmospheric channel. This will increase the receiver beam size and thus reduce the average received power but at the same time will also decrease pointing errors and may reduce turbulence-induced signal fading due to intensity scintillations. A theoretical treatment by Banach *et al.* established that as the initial longitudinal and lateral field coherence decreases, intensity fluctuations of the observed radiation also decrease.[7] More recently, two experiments have suggested that the use of a partially coherent source beam does indeed reduce intensity scintillations at the receiver.[8,9]

Previous studies of the propagation of partially coherent fields in free space are well summarized in Ref. 10. In particular, Friberg and Sudol[11] derive expressions for the beam size and the phase front radius of curvature of a partially coherent collimated beam in free space formulated so that the beam waist coincides with the transmitter. The propagation of partially coherent optical fields in atmospheric turbulence has been considered in several studies. In Ref. 12 Belenkii and Mironov perform an asymptotic analysis of the spatial mutual coherence function of a partially coherent beam in strong turbulence and estimate the accuracy of a quadratic approximation, and in Ref. 13 Belenkii *et al.* make observations concerning the influence of the partial coherence of a light source on the beam coherence radius. In Ref. 14 Wang and Plonus derive an expression for the mutual intensity function of a partially coherent laser source for focused and collimated beams. Irradiance scintillations of a partially coherent light source are considered by Fante in Ref. 15 and by Baykal *et al.* in Ref. 16.

In this paper we explore the properties of a partially (spatially) coherent quasi-monochromatic Gaussian laser beam propagating in atmospheric turbulence. Real light sources are in fact partially coherent, and a consideration of partial coherence effects may be important when considering the propagation of optical energy through a turbulent atmosphere. Expressions for average intensity, beam size, phase front radius of curvature, and wave-

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front coherence length are obtained from an analytic expression derived for the cross-spectral density. We use a Gaussian beam wave model having a parabolic wave-front phase envelope that allows full consideration of the focusing or diverging characteristics of the laser beam. The analytic results obtained here illustrate dependence of the derived quantities on both the degree of partial coherence and the effects of atmospheric turbulence and provide a better understanding of the implications for using a partially coherent laser beam in a free-space optical communication system.

2. FREE-SPACE PROPAGATION OF A GAUSSIAN BEAM

At $z = 0$ the free-space electric field of a unit-amplitude, lowest-order paraxial Gaussian beam propagating predominantly along the z axis can be represented in the form[17,18]

$$U(r, 0) = \exp\left[-\left(\frac{1}{w_o^2} + \frac{jk}{2R_o}\right)r^2\right], \tag{1}$$

where w_o is the transmitter beam radius (beam size), R_o is the radius of curvature of the phase front, $k = 2\pi/\lambda$ is the optical wave number, and $r = (x^2 + y^2)^{1/2}$ is the transversal distance from the beam center; for simplicity we introduce the notation $r^2 = |\mathbf{r}|^2$. The propagation geometry is shown in Fig. 1.

After propagating a distance z from the transmitter, the optical field becomes[19]

$$U(\rho, z) = \frac{\exp(jkz)}{\hat{r} + j\hat{z}} \exp\left[-\frac{1}{\hat{r} + j\hat{z}}\left(\frac{1}{w_o^2} + \frac{jk}{2R_o}\right)\rho^2\right], \tag{2}$$

where $\rho = (x^2 + y^2)^{1/2}$ is the transversal distance from the beam center in the receiver plane. In Eq. (2) we have used the transmitter beam parameters[19–21]

$$\hat{r}(z) = \frac{R_o - z}{R_o}, \qquad \hat{z} = \frac{z}{\hat{z}_d}, \tag{3}$$

where the normalized focusing parameter $\hat{r}$ characterizes focusing properties of the beam in terms of deviation of the wave-front curvature from the condition of optimal focusing $R_o = z$, and $\hat{z}_d = kw_o^2/2$ is diffractive distance. In this notation convergent (focused) beams are indicated by positive R_o and divergent beams by negative R_o. At the beam waist, R_o is infinite and the beam size takes its smallest value, designated as the beam waist size w_b. In terms of the focusing parameter, it follows that convergent beams are defined by $\hat{r} < 1$, collimated beams by $\hat{r} = 1$, and divergent beams by $\hat{r} > 1$. In Eqs. (3) and throughout this paper, the functional dependence of $\hat{r}$ and $\hat{z}$ on z is to be understood.

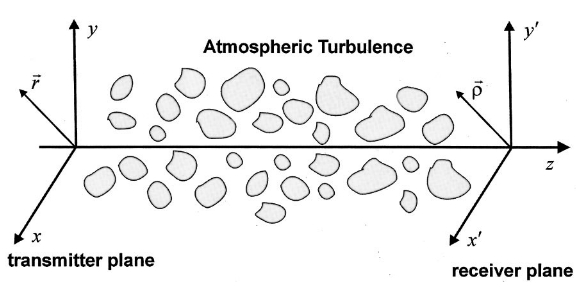

Fig. 1. Propagation geometry.

The beam size $w(z)$ and the phase front radius of curvature $R(z)$ at the receiver plane are expressed in terms of the transmitter beam parameters as

$$w(z) = w_o(\hat{r}^2 + \hat{z}^2)^{1/2}, \qquad R(z) = \frac{z(\hat{r}^2 + \hat{z}^2)}{\hat{r}(1 - \hat{r}) - \hat{z}^2}. \tag{4}$$

For a collimated beam with the waist at $z = 0$, $\hat{r} = 1$ and we obtain the well-known expressions describing beam size and phase front radius of curvature as a function of z:

$$w(z) = w_o[1 + (\lambda z/\pi w_o^2)]^{1/2},$$
$$R(z) = z[1 + (\pi w_o^2/\lambda z)]^{1/2}. \tag{5}$$

In terms of the beam size $w(z)$, the average intensity at the receiver is expressed as

$$I(\rho, z) = \frac{w_o^2}{w^2(z)} \exp\left[\frac{-2\rho^2}{w^2(z)}\right]. \tag{6}$$

Other relations involving these parameters are discussed in Ref. 20. Since it was first proposed, this beam wave model has been used in a number of studies concerning laser propagation through random media.[23,24]

3. EXTENDED HUYGENS–FRESNEL PRINCIPLE

The complex field a distance z from the transmitter can be represented, using the Huygens–Kirchhoff principle, as

$$U(\boldsymbol{\rho}, z) = \iint d^2\mathbf{r}\, G(\mathbf{r}, \boldsymbol{\rho}, z) U(\mathbf{r}, 0), \tag{7}$$

where $U(\mathbf{r}, 0)$ is the field at the transmitter plane $z = 0$. The paraxial (parabolic) wave equation derived from the Helmholtz equation has the Green's-function solution

$$G(\mathbf{r}, \boldsymbol{\rho}, z) = \frac{-jk}{2\pi z} \exp\left[jkz + \frac{jk}{2z}|\boldsymbol{\rho} - \mathbf{r}|^2 + \Psi(\mathbf{r}, \boldsymbol{\rho})\right], \tag{8}$$

where $\Psi(\mathbf{r}, \boldsymbol{\rho})$ represents the random part of the complex phase of a spherical wave due to propagation in a turbulent medium. Using this Green's-function solution, we can express Eq. (7) in terms of the Huygens–Fresnel integral[25,26]:

$$U(\boldsymbol{\rho}, z) = \frac{-jk}{2\pi z} \exp(jkz) \iint d^2\mathbf{r}\, U(\mathbf{r}, 0) \times \exp\left[\frac{jk}{2z}|\boldsymbol{\rho} - \mathbf{r}|^2 + \Psi(\mathbf{r}, \boldsymbol{\rho})\right]. \tag{9}$$

The Huygens–Fresnel principle states that each point on a wave front generates a spherical wave and that the envelope of these spherical waves constitutes a new wave front. The expression given in Eq. (9), which includes the effects of atmospheric turbulence on propagation through the quantity $\Psi(\mathbf{r}, \boldsymbol{\rho})$, is commonly called the extended Huygens–Fresnel principle and is applicable to both weak and strong turbulence fluctuation regimes.

4. CROSS-SPECTRAL DENSITY FUNCTION FOR A PARTIALLY COHERENT LASER BEAM IN TURBULENCE

Historically, wave propagation through atmospheric turbulence has been characterized in terms of the complex mutual coherence function $\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \tau) = \langle E(\boldsymbol{\rho}_1; \tau)E^*(\boldsymbol{\rho}_2; t + \tau)\rangle$, where $\langle\cdot\rangle$ denotes ensemble averaging and E is the optical frequency electric field. In contrast, recent treatments[10,11] of the propagation of partially coherent fields have focused on the cross-spectral density function $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \nu) = \langle U(\boldsymbol{\rho}_1; \nu)U^*(\boldsymbol{\rho}_2; \nu)\rangle$, which is the temporal Fourier transform of $\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \tau)$. The cross-spectral density function, which obeys the Helmholtz equation, is a measure of the correlation between the fluctuations of two field components at the same frequency. If the field is strictly monochromatic or sufficiently narrow band, so that

$$\frac{|\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1|}{c} \ll \frac{1}{\Delta\nu}, \tag{10}$$

then both characterizations yield identical results (see Appendix A). In a laser communication system a typical value for $|\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1|$ is on the order of 10 cm, which requires that $\Delta\nu < 1$ GHz.

Here we consider the behavior of $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \nu = \nu_o)$, where ν_o is the center frequency of a quasi-monochromatic Gaussian laser beam and the dependence on ν_o is to be understood. The cross-spectral density $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z)$ at the receiver can be represented as[10,26]

$$\begin{aligned} W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) &= \langle U(\boldsymbol{\rho}_1, z)U^*(\boldsymbol{\rho}_2, z)\rangle \\ &= \frac{1}{(\lambda z)^2}\iiiint d^2\mathbf{r}_1 d^2\mathbf{r}_2 W(\mathbf{r}_1, \mathbf{r}_2, 0) \\ &\quad \times \langle \exp[\Psi(\mathbf{r}_1, \boldsymbol{\rho}_1) + \Psi^*(\mathbf{r}_2, \boldsymbol{\rho}_2)]\rangle \\ &\quad \times \exp\left\{\frac{jk}{2z}[(\boldsymbol{\rho}_1 - \mathbf{r}_1)^2 - (\boldsymbol{\rho}_2 - \mathbf{r}_2)^2]\right\}. \end{aligned} \tag{11}$$

In Eq. (11) the quantity $W(\mathbf{r}_1, \mathbf{r}_2, 0)$ is the cross-spectral density at the transmitter, which we obtain by using a method based upon an approach first proposed by Schell.[27] Consider that a phase diffuser is placed over the laser transmitter aperture, so that the emitted field can be modeled as

$$\tilde{U}(\mathbf{r}, 0) = U(\mathbf{r}, 0)\exp[j\varphi_d(\mathbf{r})], \tag{12}$$

where the deterministic field $U(\mathbf{r}, 0)$ is given by Eq. (1) and the quantity $\exp[j\varphi_d(\mathbf{r})]$ represents the small random perturbation introduced by the phase diffuser.

If we assume that the ensemble average of the spatially dependent random phases introduced by the diffuser is Gaussian and depends only on the separation distance and not on the actual location on the diffuser, the cross-spectral density at the transmitter can be expressed in the form of a Gaussian Schell-model beam:

$$\begin{aligned} W(\mathbf{r}_1, \mathbf{r}_2, 0) &= \langle \tilde{U}(\mathbf{r}_1, 0)\tilde{U}^*(\mathbf{r}_2, 0)\rangle \\ &= U(\mathbf{r}_1, 0)U^*(\mathbf{r}_2, 0) \\ &\quad \times \langle \exp[j\varphi_{d1}(\mathbf{r}_1)]\exp[-j\varphi_{d2}(\mathbf{r}_2)]\rangle \\ &= U(\mathbf{r}_1, 0)U^*(\mathbf{r}_2, 0)\exp\left[\frac{-(\mathbf{r}_1 - \mathbf{r}_2)^2}{2\sigma_g^2}\right]. \end{aligned} \tag{13}$$

In Eq. (13) the quantity σ_g^2 is the variance of the Gaussian describing the ensemble average of the random phases. The partial coherence properties of the transmitter source are described by the variance σ_g^2, which is determined by the characteristics of the diffuser. Note that when σ_g^2 is infinite, the cross-spectral density is completely described by the deterministic field.

Using the sum and difference vector notation

$$\mathbf{r}_S = \tfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2), \qquad \mathbf{r}_d = \mathbf{r}_1 - \mathbf{r}_2,$$

$$\boldsymbol{\rho}_S = \tfrac{1}{2}(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2), \qquad \boldsymbol{\rho}_d = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2,$$

we can express the cross-spectral density at the transmitter as

$$\begin{aligned} W(\mathbf{r}_S, \mathbf{r}_d, 0) = \exp\Bigg\{ &-\frac{1}{w_o^2}\left[\frac{1}{2}(r_d^2 + 4r_S^2)\right] \\ &- \frac{jk}{2R_o}(2\mathbf{r}_d \cdot \mathbf{r}_S) - \frac{r_d^2}{2\sigma_g^2}\Bigg\}, \end{aligned} \tag{14}$$

and we can also write

$$\begin{aligned} &\exp\left\{\frac{jk}{2z}[(\boldsymbol{\rho}_1 - \mathbf{r}_1)^2 - (\boldsymbol{\rho}_2 - \mathbf{r}_2)^2]\right\} \\ &\quad = \exp\left\{\frac{jk}{z}[(\mathbf{r}_S - \boldsymbol{\rho}_S) \cdot (\mathbf{r}_d - \boldsymbol{\rho}_d)]\right\}. \end{aligned} \tag{15}$$

Yura[28] shows that the random part of the complex phase of a spherical wave propagating in homogeneous turbulence can be approximated by

$$\begin{aligned} &\langle \exp(\Psi(\mathbf{r}_1, \boldsymbol{\rho}_1) + \Psi^*(\mathbf{r}_2, \boldsymbol{\rho}_2))\rangle \\ &\quad \cong \exp\left[\frac{-1}{\rho_o^2}(r_d^2 + \mathbf{r}_d \cdot \boldsymbol{\rho}_d + \rho_d^2)\right], \end{aligned} \tag{16}$$

where $\rho_o(z) = (0.55C_n^2k^2z)^{-3/5}$ is the coherence length of a spherical wave propagating in turbulence and C_n^2 is the refractive-index structure parameter describing the strength of atmospheric turbulence. We can now express the cross-spectral density at the field point:

$$W(\boldsymbol{\rho}_S,\ \boldsymbol{\rho}_d,\ z) = \frac{1}{(\lambda z)^2}\iint d^2\mathbf{r}_d \iint d^2\mathbf{r}_S \exp\left(\frac{-2r_S^2}{w_o^2}\right) \times \exp\left[\frac{-jk\mathbf{r}_S\cdot\mathbf{r}_d}{R_o} + \frac{jk\mathbf{r}_S\cdot(\mathbf{r}_d-\boldsymbol{\rho}_d)}{z}\right] \times \exp\left[-\frac{r_d^2}{2w_o^2} - \frac{r_d^2}{2\sigma_g^2} - \frac{r_d^2+\mathbf{r}_d\cdot\boldsymbol{\rho}_d+\rho_d^2}{\rho_o^2} - \frac{jk\boldsymbol{\rho}_S\cdot(\mathbf{r}_d-\boldsymbol{\rho}_d)}{z}\right]. \tag{17}$$

Evaluating this integral, we obtain the expression for the cross-spectral density at the receiver:

$$W(\boldsymbol{\rho}_S,\ \boldsymbol{\rho}_d,\ z) = \frac{w_o^2}{w_\zeta^2(z)}\exp\left\{-\rho_d^2\left(\frac{1}{\rho_o^2}+\frac{1}{2w_o^2\hat{z}^2}\right) + \frac{2j\boldsymbol{\rho}_S\cdot\boldsymbol{\rho}_d}{w_o^2\hat{z}}\right\}\exp\left[\frac{-2\rho_S^2}{w_\zeta^2(z)}\right] \times \exp\left[\frac{-(j\phi)^2\rho_d^2}{2w_\zeta^2(z)}\right]\exp\left[\frac{-2j\phi\,\boldsymbol{\rho}_S\cdot\boldsymbol{\rho}_d}{w_\zeta^2(z)}\right],$$

$$\phi \equiv \frac{\hat{r}}{\hat{z}} - \hat{z}\frac{w_o^2}{\rho_o^2}. \tag{18}$$

A. Beam Size

In Eq. (18) we have defined the beam size (radius) $w_\zeta(z)$ of a partially coherent beam in turbulence and the global coherence parameter ζ as

$$w_\zeta(z) = w_o(\hat{r}^2 + \zeta\hat{z}^2)^{1/2}, \qquad \zeta = 1 + \frac{w_o^2}{\sigma_g^2} + \frac{2w_o^2}{\rho_o^2}. \tag{19}$$

Physically speaking, the global coherence parameter is a measure of the global degree of coherence of light across each transverse plane along the propagation path. For a coherent beam ($\sigma_g \to \infty$) in the absence of atmospheric turbulence ($\rho_o \to \infty$), the global coherence parameter reduces to unity and the beam size reduces to its diffractive equivalent given in Eqs. (4). Note that the global coherence parameter ζ is a function of the path length z through the coherence length of a spherical wave propagating in turbulence, given by $\rho_o(z) = (0.55C_n^2k^2z)^{-3/5}$.

It is also useful to define a related dimensionless quantity, the source coherence parameter, as

$$\zeta_S = 1 + \frac{w_o^2}{\sigma_g^2}. \tag{20}$$

The degree of partial (spatial) coherence of the source laser beam at the transmitter is completely specified by the parameter ζ_S. A quantity similar to the source coherence parameter ζ_S was defined earlier by Friberg and Sudol[11] and later used by Mandel and Wolf[10] to describe partial coherence properties of the source beam.

It was shown in Ref. 29 that an earlier derivation for the size of a partially coherent beam in turbulence[14] contains an error. However, we can compare the expression in Eq. (19) with the expression for a partially coherent collimated beam in free space given in Refs. 10 and 11. For a collimated beam ($\hat{r} = 1, \hat{z} = z/0.5kw_b^2$) with the beam waist w_b located at the $z = 0$ (transmitter) plane, Eq. (19) takes the form

$$w_\zeta(z) = w_b(1 + \zeta_S\hat{z}^2)^{1/2} = w_b\left[1 + \left(\frac{2z}{k\,\delta w_b}\right)^2\right]^{1/2}, \tag{21}$$

where $1/\delta^2 = 1/4\sigma_S^2 + 1/\sigma_g^2$ is defined in Ref. 10. If we equate the square of the transmitter beam size (beam waist) with two standard deviations of the Gaussian field ($w_b^2 = 4\sigma_S^2$), it follows that

$$w_\zeta(z) = w_b\Delta(z) = 2\sigma_S\Delta(z), \qquad \Delta(z) = \left[1 + \left(\frac{2z}{k\,\delta w_b}\right)^2\right]^{1/2}, \tag{22}$$

where $\Delta(z)$ is the expansion coefficient of the beam. Thus when free-space propagation is assumed, the beam size obtained here for a partially coherent beam in atmospheric turbulence exactly reduces to its free-space equivalent obtained in Ref. 10.

Figure 2 shows the normalized beam size $w_\zeta(z)/w_o^2$ as a function of the normalized distance $\hat{z}$ for values of the source coherence parameter ζ_S representing beams from the coherent ($\zeta_S = 1$) to the partially coherent ($\zeta_S = 26$). As the global coherence parameter increases past unity, the spatial coherence of the beam decreases and the beam begins to increase beyond its diffractive size. However, the beam still retains to some extent its ability to focus, although this ability decreases as coherence is lost.

Figure 3 compares behavior of the coherent beam with that of the partially coherent beam ($\zeta_S = 26$) for $\hat{r} = 0.001$ (convergent beam) and $\hat{r} = 1$ (collimated beam). The partially coherent beams behave like the coherent beams, except that the expected increase in normalized beam size occurs at a smaller value of $\hat{z}$, with the magnitude of this shift being a function of the size of the source coherence parameter ζ_S. A similar shift of the beam waist toward the transmitting aperture was noted earlier

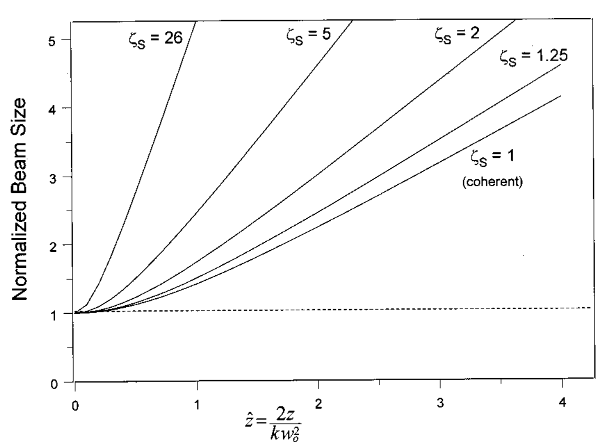

Fig. 2. Normalized beam size $w_\zeta^2(z)/w_o^2$ as a function of normalized distance $\hat{z} = z/\hat{z}_d$ for different values of the source coherence parameter ζ_S.

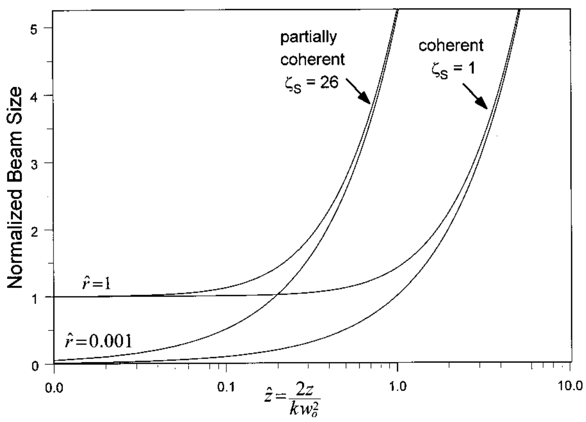

Fig. 3. Normalized beam size $w_\zeta^2(z)/w_o^2$ as a function of normalized distance $\hat{z} = z/\hat{z}_d$ for a coherent ($\zeta_S = 1$) and a partially coherent ($\zeta_S = 26$) collimated beam ($\hat{r} = 1$) and for a coherent and a partially coherent beam with focusing ($\hat{r} = 0.001$).

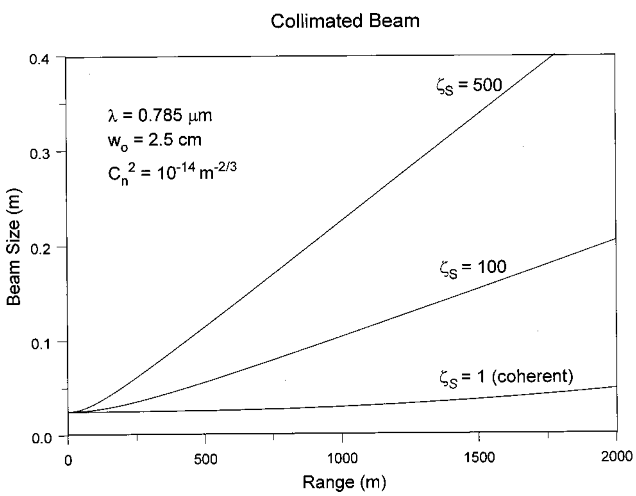

Fig. 4. Comparison of the beam size for a coherent collimated beam with beam sizes for two partially coherent collimated beams with different source coherences in moderate turbulence after propagating 2 km.

for coherent beams propagating in turbulence.[30] In Ref. 30 the magnitude of this shift was dependent on the strength of turbulence and could potentially be on the order of hundreds of meters or more for strong turbulence. Here we find that for a partially coherent source beam the magnitude of this shift is dependent on the degree of spatial coherence.

Equation (19) offers a method for calculating the desired transmitter diffuser characteristics in order to obtain the optimal beam footprint at the receiver for given turbulence conditions and thus reduce pointing errors. After propagating 2 km through moderate atmospheric turbulence, a collimated beam ($\lambda = 0.785\ \mu$m) with a transmitter beam size of 2.5 cm will have a beam footprint of only 9.6 cm. With such a small beam footprint, the probability of pointing errors is very high. By placing a phase diffuser directly in front of the laser transmitter, one can increase the beam size at the receiver to accommodate receiver motion due to tower or building sway, beam motion due to atmospheric turbulence effects, etc. As shown in Fig. 4, if we choose $\zeta_S = 100$, the beam footprint after propagation of 2 km through moderate atmospheric turbulence will be 41 cm, and for $\zeta_S = 500$ the beam footprint will be 90 cm. However, since increasing the beam footprint also reduces the power incident on a fixed-size receiver, it is important not to overestimate the receiver beam size necessary to reduce pointing errors.

B. Average Intensity

The average intensity $\langle I(\boldsymbol{\rho})\rangle$ for a unit-amplitude beam is obtained from Eq. (18) when $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2$, so that $\boldsymbol{\rho}_S = \boldsymbol{\rho}$ and $\boldsymbol{\rho}_d = 0$:

$$\langle I(\rho)\rangle = \frac{w_o^2}{w_\zeta^2(z)} \exp\left[\frac{-2\rho^2}{w_\zeta^2(z)}\right]. \tag{23}$$

Note that the combined effects of partial coherence and turbulence operate solely on the normalized distance $\hat{z}$ through the beam size $w_\zeta(z)$. The expression given in Eq. (23) for the average intensity exactly reduces to the expression for a coherent beam in free space given in Eq. (6).

Using the notation of Ref. 10, define the 1/*e* falloff in intensity as $\bar{\rho}_S(z)$. Recalling Eq. (22), write

$$\bar{\rho}_S(z) = \frac{w_\zeta(z)}{\sqrt{2}} = \frac{2\sigma_S\Delta(z)}{\sqrt{2}} = \sqrt{2}\sigma_S\Delta(z), \tag{24}$$

which exactly corresponds to the result given in Ref. 10 for a partially coherent collimated beam propagating in free space.

C. Phase Front Radius of Curvature

The phase front radius of curvature for a partially coherent laser beam propagating in atmospheric turbulence is obtained from the complex portion of Eq. (18):

$$\{W(\boldsymbol{\rho}_S, \boldsymbol{\rho}_d, z)\} = \frac{w_o^2}{w_\zeta^2(z)} \exp\left[\frac{-jk\boldsymbol{\rho}_S \cdot \boldsymbol{\rho}_d}{R_\zeta(z)}\right], \tag{25}$$

where the radius of curvature $R_\zeta(z)$ is defined as

$$R_\zeta(z) = \frac{z(\hat{r}^2 + \zeta\hat{z}^2)}{\phi\hat{z} - \zeta\hat{z}^2 - \hat{r}^2}, \qquad \phi \equiv \frac{\hat{r}}{\hat{z}} - \hat{z}\frac{w_o^2}{\rho_o^2}. \tag{26}$$

For a coherent beam in free space, this expression for the radius of curvature reduces to its diffractive form given in Eqs. (4). Equation (26) also exactly reduces to the expression for a partially coherent collimated beam propagating in free space given in Ref. 11.

The normalized radius of curvature for a collimated beam is shown in Fig. 5 as a function of normalized distance $\hat{z}$ for different values of the source coherence parameter ζ_S. For a diffractive beam in free space the radius of curvature is infinite at $\hat{z} = 0$ since the beam waist is collocated with the transmitter, and again approaches infinity when $\hat{z}$ becomes large. As ζ_S increases, indicating a less coherent source beam, observe the strong focusing [pronounced dip in $R_\zeta(z)$] that occurs for small values of $\hat{z}$. For values of $\hat{z} > 2$ there is an increasingly diminished effect on the radius of curvature due to partial coherence.

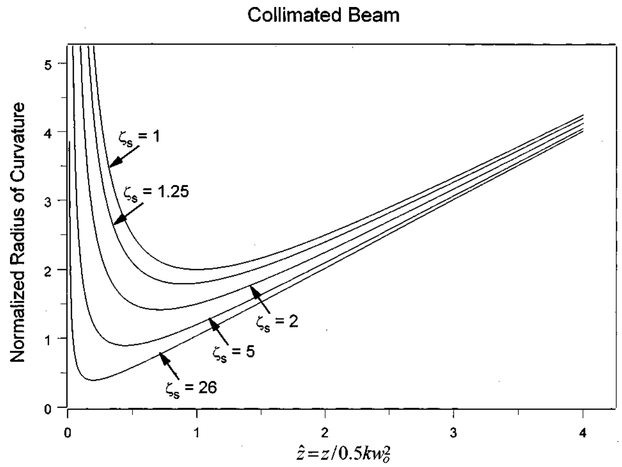

Fig. 5. Normalized radius of curvature $R_\zeta(z)/(0.5w_o^2k)$ as a function of normalized distance $\hat{z}$ for different values of ζ_S.

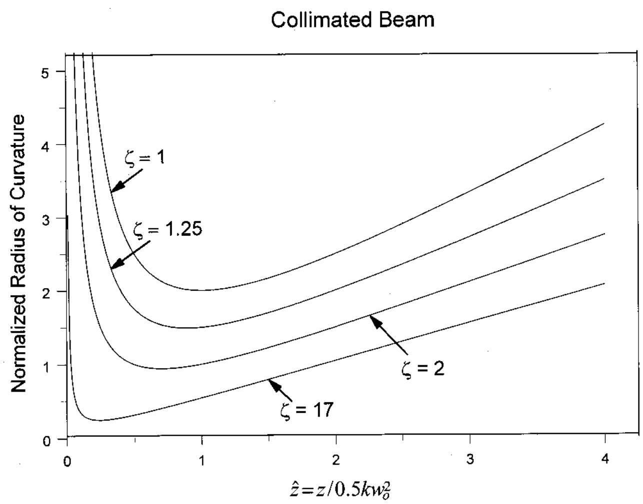

Fig. 6. Normalized radius of curvature $R_\zeta(z)/(0.5w_o^2k)$ as a function of normalized distance $\hat{z}$ for varying strengths of atmospheric turbulence. The source coherence parameter $\zeta_S = 1$ for each curve, so that any partial coherence effects are due to atmospheric turbulence.

While in Fig. 5 we assume that any increase in global coherence is due to a less coherent source, in Fig. 6 we specifically consider the role of atmospheric turbulence by setting $\zeta_S = 1$. Note that when atmospheric turbulence is present, the radius of curvature displays increased focusing for all values of $\hat{z}$, not just for values of $\hat{z} < 2$. Compare the curve for $\zeta = 26$ in Fig. 5 with the curve for $\zeta = 17$ in Fig. 6, both of which have an equivalently sharp minimum in the radius of curvature occurring at approximately $\hat{z} = 0.3$. Through the influence of ρ_o on the auxiliary beam parameter ϕ, the radius of curvature is more strongly affected by turbulence than is the beam size. In spite of the different global coherence parameter values, the curves in Figs. 5 and 6 are quite similar except at higher values of $\hat{z}$, where because of the influence of turbulence the radius of curvature does not eventually approach its diffractive value, as it does in Fig. 5.

D. Wave-Front Coherence

To obtain the complex degree of coherence of the partially coherent optical wave, consider the normalized quantity[10,14]

$$\mu(\rho_d, z) = \frac{W(0, \boldsymbol{\rho}_d, z)}{W(0, 0, z)} = \exp\left[-\rho_d^2\left(\frac{1}{\rho_o^2} + \frac{1}{2w_o^2\hat{z}^2}\right)\right]\exp\left[\frac{-(j\phi)^2\rho_d^2}{2w_\zeta^2(z)}\right]. \tag{27}$$

After simplification we obtain an analytic expression for the complex degree of coherence $\mu(\rho_d, z)$:

$$\mu(\rho_d, z) = \exp\left[-\frac{\rho_d^2}{\rho_o^2}\left(1 + \frac{\rho_o^2}{2w_o^2\hat{z}^2} - \frac{\phi^2\rho_o^2}{2w_\zeta^2(z)}\right)\right] = \exp\left(-\frac{\rho_d^2}{\rho_C^2}\right). \tag{28}$$

In Eq. (28) ρ_C is the coherence length of the optical field at the receiver and is given by

$$\rho_C = \rho_o\left(1 + \frac{\rho_o^2}{2w_o^2\hat{z}^2} - \frac{\phi^2\rho_o^2}{2w_\zeta^2(z)}\right)^{-1/2}. \tag{29}$$

An expression given earlier in Ref. 14 for the coherence length of a partially coherent beam in turbulence contains an error.

The effect of a partially coherent source beam on the wave-front coherence length ρ_C is illustrated in Fig. 7, where coherence length is shown as a function of atmospheric turbulence strength. As the source beam becomes less coherent, the wave-front coherence length decreases as expected. For each value of source coherence, the coherence length maintains an almost constant value until turbulence strength increases to a point where atmospheric turbulence effects dominate wave-front coherence. This condition is described by the hypothetical line

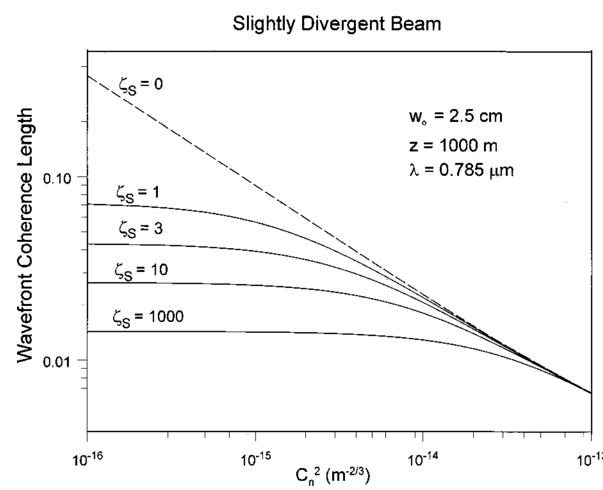

Fig. 7. Wave-front coherence length ρ_C as a function of the refractive-index structure parameter C_n^2 for a slightly divergent beam ($\hat{r} = 2$) showing the effects of having a partially coherent source beam: $\zeta_S = 1$ (coherent source beam), 3, 10, and 1000. The hypothetical line $\zeta_S = 0$ is shown as an upper bound.

determined by $\zeta_S = 0$; in the limit $\zeta_S = 0$ the global coherence parameter is completely determined by losses in coherence due to atmospheric turbulence. In short, when $\sigma_g/\rho_o \ll 1$, the effects of source coherence will dominate the global coherence parameter ζ and the wave-front coherence length decreases when source coherence is increased. Conversely, when $\sigma_g/\rho_o \gg 1$, any effects due to having a partially coherent source beam are overtaken by the loss in wave-front coherence induced by atmospheric turbulence.

We can compare the results derived here for any type of beam focusing and source coherence with those given in Ref. 10 for a partially coherent collimated beam in free space. Again using the notation of Ref. 10, define the 1/*e* falloff in coherence as $\bar{\rho}_\mu(z)$. The expression for wave-front coherence given in Eq. (29) for a collimated beam ($\hat{r}_o = 1$, $\hat{z} = 2z/kw_b^2$) in the absence of turbulence ($\phi = \hat{r}/\hat{z}$) can be expressed as

$$\frac{1}{\bar{\rho}_\mu^2(z)} = \frac{1}{2w_b^2\hat{z}^2} - \frac{\phi^2}{2w_\zeta^2(z)} = \frac{1}{2w_b^2\hat{z}^2} - \frac{1}{2w_\zeta^2(z)}. \tag{30}$$

Recalling that $w_\zeta(z) = w_b(z)\Delta(z)$ and again using the relationship $w_b^2 = 4\sigma_S^2$, we obtain

$$\bar{\rho}_\mu(z) = \sqrt{2}\,\delta\Delta(z), \tag{31}$$

which is identical to the result obtained in Ref. 10 for a partially coherent collimated beam propagating in free space.

Note that in any transverse cross section of a collimated beam in free space the ratio of the beam size to the coherence size is constant on propagation:

$$\frac{\bar{\rho}_S(z)}{\bar{\rho}_\mu(z)} = \frac{\sigma_S}{\delta} = \frac{\sqrt{\zeta_S}}{2}; \tag{32}$$

that is, the degree of global coherence of light in any transverse cross section of a Gaussian Schell-model beam is invariant on propagation. This key result was first given in Ref. 10 for a partially coherent collimated beam in free space:

$$\frac{\bar{\rho}_S(z)}{\bar{\rho}_\mu(z)} = \frac{\sigma_S}{\sigma_g}. \tag{33}$$

This invariance in the global coherence of light across any transverse cross section of the beam also exists for a partially coherent beam with specified focusing or diverging characteristics propagating in atmospheric turbulence:

$$\frac{\bar{\rho}_S(z)}{\bar{\rho}_\mu(z)} = \frac{w_\zeta^2(z)}{2\rho_o^2} + \frac{w_\zeta^2(z)}{4w_o^2\hat{z}^2} - \frac{\phi^2}{4}. \tag{34}$$

5. CONCLUDING REMARKS

We have derived an expression for the cross-spectral density function of a partially coherent quasi-monochromatic Gaussian laser beam that describes propagation in either atmospheric turbulence or free space and that allows for consideration of the focusing or diverging characteristics of the beam. From the cross-spectral density function, expressions for average intensity, beam size, phase front radius of curvature, complex degree of coherence, and wave-front coherence length were obtained. These expressions exactly reduce to their diffractive equivalents when full coherence and the absence of turbulence are assumed, and they exactly correspond to expressions obtained previously for a partially coherent collimated beam in the absence of turbulence.

The global coherence parameter, a measure of the global degree of coherence of light across each transverse plane along the propagation path, and the related source coherence parameter that describes coherence properties of the source beam at the transmitter were also defined. Beam size, phase front radius of curvature, and wave-front coherence were examined as a function of the source coherence. When $\sigma_g/\rho_o \ll 1$ source coherence effects dominate the behavior of wave-front coherence. As atmospheric turbulence becomes stronger so that $\sigma_g/\rho_o \gg 1$, source coherence has little effect on wave-front coherence and atmospheric turbulence strength drives the behavior of wave-front coherence. It is also shown that the global coherence of light across any transverse cross section of the beam is invariant for partially coherent beams in atmospheric turbulence, regardless of focusing or diverging characteristics.

A companion paper is being completed that augments the results obtained here and demonstrates that the average bit error rate in a free-space laser communication system is reduced when the transmitted laser beam is partially coherent. Results obtained in this companion paper also indicate that because of the quadratic approximation for the phase structure function[28] the validity of the expressions obtained here may be restricted to the weak fluctuation regime.

APPENDIX A: RELATIONSHIP BETWEEN THE MUTUAL COHERENCE FUNCTION AND THE CROSS-SPECTRAL DENSITY FUNCTION

The propagation geometry for the cross-spectral density function is shown in Fig. 8. When comparing Fig. 8 with Fig. 1, note that in Fig. 1 the z dependency in the vector $\boldsymbol{\rho}$ is explicitly expressed ($\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ have identical z components). From Eq. (5.3-1) in Ref. 10, the integral representation of the cross-spectral density function $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \nu)$ using the Green's-function approach is

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \nu) = \left(\frac{k}{2\pi}\right)^2 \int\int d^2\mathbf{r}_1\, d^2\mathbf{r}_2\, W(\mathbf{r}_1, \mathbf{r}_2; \nu) \times \frac{\exp[jk(R_2 - R_1)]}{R_2 R_1}, \tag{A1}$$

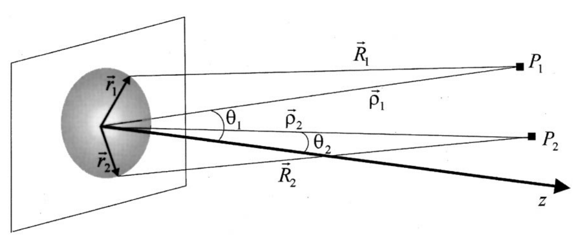

Fig. 8. Propagation geometry for the cross-spectral density function.

where the obliquity factors $\cos\theta_1$ and $\cos\theta_2$ given in Eq. (5.3-1) in Ref. 10 have been taken as unity. Next, take the temporal Fourier transform of both sides and use the fact that $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \nu)$ and $\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \tau)$ form a Fourier transform pair:

$$\int_0^\infty \mathrm{d}\nu \exp(-2\pi j\nu\tau) W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \nu) = \Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \tau) = \int_0^\infty \mathrm{d}\nu \exp(-2\pi j\nu\tau) \times \left\{\left(\frac{k}{2\pi}\right)^2 \int\int \mathrm{d}^2\mathbf{r}_1 \mathrm{d}^2\mathbf{r}_2 W(\mathbf{r}_1, \mathbf{r}_2; \nu) \times \frac{\exp[jk(R_2 - R_1)]}{R_2 R_1}\right\}. \tag{A2}$$

Use the fact that $k = 2\pi\nu/c$ and exchange the order of integration to obtain

$$\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \tau) = \frac{1}{R_2 R_1}\int\int \mathrm{d}^2\mathbf{r}_1 \mathrm{d}^2\mathbf{r}_2 \int_0^\infty \mathrm{d}\nu \left(\frac{\nu}{c}\right)^2 \times W(\mathbf{r}_1, \mathbf{r}_2; \nu)\exp(-2\pi j\nu\tau) \times \exp\left[\frac{2\pi j\nu(R_2 - R_1)}{c}\right]. \tag{A3}$$

If we assume that the cross-spectral density of the source is "narrow band," i.e., it is nonzero over a small enough range of frequencies that we can make the approximation

$$\frac{\nu}{c} \cong \frac{\bar{k}}{2\pi},$$

where ν is frequency, c is the speed of light in a vacuum, and $\bar{k}$ is the central wave number of the frequency band, then we can write

$$\int_0^\infty \mathrm{d}\nu\, W(\mathbf{r}_1, \mathbf{r}_2; \nu)\exp(-2\pi j\nu\tau)\exp\left[\frac{2\pi j\nu(R_2 - R_1)}{c}\right] = \Gamma\left(\mathbf{r}_1, \mathbf{r}_2; \tau - \frac{R_2 - R_1}{c}\right) \tag{A4}$$

by the properties of Fourier transforms. If we use the properties of narrow-band analytic signals, we can further write (see Sec. 4.4.3 in Ref. 10)

$$\Gamma\left(\mathbf{r}_1, \mathbf{r}_2; \tau - \frac{R_2 - R_1}{c}\right) \approx \Gamma(\mathbf{r}_1, \mathbf{r}_2; \tau)\exp[j\bar{k}(R_2 - R_1)]. \tag{A5}$$

Using Eq. (A4) and substituting approximation (A5) into integral (A3) yields the final result:

$$\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \tau) = \left(\frac{\bar{k}}{2\pi}\right)^2 \int\int \mathrm{d}^2\mathbf{r}_1 \mathrm{d}^2\mathbf{r}_2 \Gamma(\mathbf{r}_1, \mathbf{r}_2; \tau) \times \frac{\exp[j\bar{k}(R_2 - R_1)]}{R_2 R_1}. \tag{A6}$$

Thus the mutual coherence function and the cross-spectral density yield identical results as long as

$$\frac{R_2 - R_1}{c} \ll \frac{1}{\Delta\nu} \tag{A7}$$

or, equivalently,

$$\frac{|\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1|}{c} \ll \frac{1}{\Delta\nu}. \tag{A8}$$

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